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Superadditivity of Wigner-Yanase-Dyson Information Revisited

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Abstract The Wigner-Yanase-Dyson information is an important variant of quantum Fisher information. Two fundamental requirements concerning this notion of information content originally postulated by Wigner and Yanase are convexity and superadditivity. The former was fully established by Lieb in 1973, and led to the first proof of the strong subadditivity of quantum entropy. The latter, although widely believed to be true, was quite recently disproved by Hansen. Nevertheless, superadditivity has also been established in two extreme cases, i.e., when the states are pure or classical. In this paper, we first review a scheme to classify bipartite states into a hierarchy of classical, semi-quantum, and quantum states, which are arranged in the order of increasing quantumness. We then prove the superadditivity of the Wigner-Yanase-Dyson information for all semi-quantum states. The convexity of the Wigner-Yanase-Dyson information plays a crucial role here.

Keywords Wigner-Yanase-Dyson information · Convexity · Superadditivity · Semi-quantum state · Local quantum measurement

1 Introduction

The quantification of information content of an object is a fundamental issue in a variety of scientific disciplines such as communications theory, statistics, and physics. In classical physics, a state is represented by a probability distribution and an observable is represented by a random variable, and measurements can always be performed without disturbing the

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system, at least in principle. Two prototypical classical information concepts are the Shannon entropy [21] and the Fisher information [5]. In quantum physics, a state is mathematically represented by a density operator and an observable is represented by a self-adjoint operator. Here a radical difference from the classical scenario is that a generic measurement usually disturbs a system, unless the system happens to be essentially classical. As direct quantum extensions of the corresponding classical notions, quantum entropy [23] and various quantum extensions of Fisher information are now widely studied and used in quantum information theory [8, 9, 14, 19].

In this paper, we are interested in a particular and celebrated variant of quantum Fisher information, i.e., the Wigner-Yanase-Dyson information. This important quantity has its origin in the study of quantum measurement theory by Wigner and Yanase in 1963 [25]. The so-called skew information,

$$I(\rho, H) = -\frac{1}{2}\mathrm{tr}[\rho^{1/2}, H]^2$$

was first proposed by them as the amount of information content in ρ (with respect to H). Here tr denotes trace, the square bracket denotes commutator (that is, $[A, B] = AB - BA$), ρ is a density operator, and H is a fixed self-adjoint operator serving as a conserved observable. The original intention of Wigner and Yanase is to quantify the general observation that an observable not commuting with the conserved observable H is more difficult to measure than those commuting with H, and they used the skew information to quantify the amount of information on the values of observables *not* commuting with (i.e., skew to) H. Alternatively, the skew information can be more directly interpreted as an intuitive variant of quantum Fisher information [14]. As mentioned by Wigner and Yanase [25], Dyson suggested the following more general quantity

$$I_\gamma(\rho, H) = -\frac{1}{2}\mathrm{tr}[\rho^\gamma, H][\rho^{1-\gamma}, H], \quad 0 < \gamma < 1,$$

which is now called the Wigner-Yanase-Dyson information, as a measure of information content of ρ with respect to H. Clearly, $I_{1/2}(\rho, H)$ is just the skew information.

Wigner and Yanase discussed some fundamental requirements for $I_\gamma(\rho, H)$ to be a good measure of information. The most important ones are convexity and superadditivity. The convexity conjecture states that

$$I_\gamma(\lambda_1\rho_1 + \lambda_2\rho_2, H) \le \lambda_1 I_\gamma(\rho_1, H) + \lambda_2 I_\gamma(\rho_2, H),$$

where $\lambda_1 + \lambda_2 = 1$, $\lambda_i \ge 0$, ρ_1 and ρ_2 are density operators, and H is an observable. The physical meaning of convexity is that when two states are mixed, their information content should decrease because one then cannot tell which from which and thus loses the identity information.

The convexity for the case $\gamma = 1/2$ is first proved by Wigner and Yanase themselves [25], and for the general case of any $\gamma \in (0, 1)$, known as the celebrated Wigner-Yanase-Dyson conjecture, was remarkably established by Lieb in 1973 [12]. Lieb's convexity theorem not only trigged further research and several different proofs [2, 4, 6, 11, 22], but also led to the first proof of the strong subadditivity of quantum entropy and the monotonicity of quantum relative entropy [13, 22], which are fundamental and extremely important results [10].

The superadditivity conjecture states that for any bipartite density operator ρ shared by parties a and b,

$$I_\gamma(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) \ge I_\gamma(\rho_a, H_a) + I_\gamma(\rho_b, H_b). \tag{1}$$

Here $\rho_a = \mathrm{tr}_b \rho$, $\rho_b = \mathrm{tr}_a \rho$ (partial trace) are the marginal states, H_a and H_b are observables, for parties a and b, respectively, $\mathbf{1}$ is the identity operator (depending on the system), and $\otimes$ denotes tensor product of operators. The plausible physical significance of superadditivity lies in that when a composite system is separated into two parties, the information content should, in general, drop, because the correlations between the two parties are lost.

The above inequality (1) was suggested by Wigner and Yanase as a necessary [25], and by Lieb as an absolute [12], requirement, for $I_\gamma(\rho, H)$ to be a satisfying measure of information content. Unfortunately, this longstanding conjecture is false in general, as illustrated by a remarkable numerical counterexample of Hansen [7]. Nevertheless, the superadditivity has also been established in two extreme and significant cases, i.e., when ρ is pure or classical [15], and two weak forms of superadditivity have also been established [1].

Though the superadditivity conjecture is false, because of its importance, it is still desirable to identify conditions as general as possible to rescue the superadditivity. In this paper, we will establish the superadditivity for any semi-quantum bipartite states, i.e., those states which are essentially classical on one party and quantum on the other party. This class of states arises naturally when classifying bipartite states from a measurement perspective, and plays an important role in the Holevo theory of accessible information and the investigation of quantum discord [18].

This paper is arranged as follows. In Sect. 2, we introduce a hierarchal classification scheme for bipartite states in terms of local measurements. Three classes of states are distinguished: classical, semi-quantum, and quantum. We establish the superadditivity for semi-quantum states in Sect. 3. Finally, Sect. 4 is devoted to the summary and discussion. In this paper, we always work on finite dimensional spaces in order to avoid technical complications. The results can be straightforwardly extended to infinite dimensional cases by paying due attention to the domain issues.

2 Classifying Bipartite States via Local Measurements

Bipartite states, mathematically represented by density operators on tensor product Hilbert spaces $\mathcal{H}_a \otimes \mathcal{H}_b$, are primary objects for encoding correlations, and their structures are extremely rich and intricate. An important issue is to qualitatively classify them. For example, the entanglement/separability dichotomic paradigm, formalized by Werner [24], classifies bipartite states into entangled and separable classes. This classification has tremendous influence in the study of entanglement phenomenon and the new field of quantum information theory [17].

Here, we use a different classification in terms of local measurements for our purpose [16]. First, recall that a von Neumann measurement is a set of one-dimensional orthogonal projections, which constitutes a resolution of the identity. In this paper, by a measurement we will always mean a von Neumann measurement.

Definition 1 Let ρ be a bipartite state shared by parties a and b.

(1) If there exist a local measurement $P^a = \{P_i^a\}$ for party a and a local measurement $P^b = \{P_j^b\}$ for party b such that ρ is left invariant after the joint measurement $P^a \otimes P^b = \{P_i^a \otimes P_j^b\}$, i.e.,

$$\rho = \sum_{ij} (P_i^a \otimes P_j^b)\rho(P_i^a \otimes P_j^b),$$

then ρ is called classical.

(2) If there exists a local measurement $P = \{P_i\}$ for party a such that ρ is left invariant under it, i.e.,

$$\rho = \sum_i (P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1}),$$

then ρ is called semi-quantum. Of course, we can also interchange the role of a and b, but for simplicity, we will take this convention.

(3) To make the comparison convenient, any bipartite states, including the classical and semi-quantum ones, will be called, *a priori*, quantum.

From the above definition, we have the following hierarchy

$$\{\text{classical states}\} \subset \{\text{semi-quantum states}\} \subset \{\text{ quantum states}\}.$$

Proposition 1 *Let ρ be a bipartite state shared by parties a and b.*

(1) *The state ρ is classical if and only if there exist orthonormal bases $\{|\psi_i^a\rangle\}$ and $\{|\psi_j^b\rangle\}$ for parties a and b, respectively, such that*

$$\rho = \sum_{ij} p_{ij} |\psi_i^a\rangle\langle\psi_i^a| \otimes |\psi_j^b\rangle\langle\psi_j^b|,$$

where $\{p_{ij}\}$ is a classical bivariate probability distribution (we allow $p_{ij} = 0$ for certain i, j).

(2) *The state ρ is semi-quantum if and only if there exists an orthonormal base $\{|\psi_i\rangle\}$ for party a such that*

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes \rho_i,$$

where $\{p_i\}$ is a classical probability distribution (we allow $p_i = 0$ for certain i), and ρ_i are density operators for party b.

Proof We only prove (2). The proof of (1) is similar [16].

First, if ρ can be expressed in the above form, then put $P_i = |\psi_i\rangle\langle\psi_i|$, we see readily that ρ is left invariant under the local measurement $P = \{P_i\}$.

Conversely, suppose that ρ is invariant under a local measurement $P = \{P_i\}$, that is,

$$\rho = P(\rho),$$

where

$$P(\rho) = \sum_i (P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1}),$$

then noting that $(P_i \otimes \mathbf{1})(P_j \otimes \mathbf{1}) = \delta_{ij} P_i \otimes \mathbf{1}$, we have

$$\begin{aligned} P(\rho^2) &= P(\rho P(\rho)) \\ &= \sum_i (P_i \otimes \mathbf{1})\rho P(\rho)(P_i \otimes \mathbf{1}) \\ &= \sum_{ij} (P_i \otimes \mathbf{1})\rho(P_j \otimes \mathbf{1})\rho(P_j \otimes \mathbf{1})(P_i \otimes \mathbf{1}) \end{aligned}$$

$$= \sum_i (P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1}).$$

On the other hand,

$$\begin{aligned}\rho^2 &= (P(\rho))^2 \\ &= \left(\sum_i (P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1})\right)^2 \\ &= \sum_{ij} (P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1})(P_j \otimes \mathbf{1})\rho(P_j \otimes \mathbf{1}) \\ &= \sum_i (P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1}).\end{aligned}$$

Therefore, we have proved that $P(\rho^2) = \rho^2$. From this identity, we have

$$\begin{aligned}&\sum_i [\rho, P_i \otimes \mathbf{1}][\rho, P_i \otimes \mathbf{1}]^\dagger \\ &\quad = \sum_i \big(\rho(P_i \otimes \mathbf{1})^2\rho - \rho(P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1}) - (P_i \otimes \mathbf{1})\rho(P_i \otimes \mathbf{1})\rho + (P_i \otimes \mathbf{1})\rho^2(P_i \otimes \mathbf{1})\big) \\ &\quad = \rho^2 - \rho P(\rho) - P(\rho)\rho + P(\rho^2) \\ &\quad = 0,\end{aligned}$$

from which we conclude that $[\rho, P_i \otimes \mathbf{1}] = 0$, that is, $P_i \otimes \mathbf{1}$ commutes with ρ for any i. This in turn implies the ρ can be written as

$$\rho = \sum_i p_i P_i \otimes \rho_i$$

for some probability distribution $\{p_i\}$ and density operators ρ_i for party b. Because $\{P_i\}$ is a von Neumann measurement, it can be expressed as $P_i = |\psi_i\rangle\langle\psi_i|$ with $\{|\psi_i\rangle\}$ constituting an orthonormal base for party a. □

Since orthogonal states can always be perfectly distinguished, by Proposition 1, we see that, from the viewpoint of the composite system, a classical state is classical on both parties a and b, and can be identified with a classical bivariate probability distribution, while a semi-quantum state is classical on party a, and may be quantum on party b. It is impossible to identify a generic semi-quantum state with a classical bivariate probability distribution unless the family of operators ρ_i commute (in such an instance, the semi-quantum state is actually classical). For a semi-quantum state, only the marginal state for party a can be identified with a classical probability distribution, the other marginal state may not have such an identification. It should be noted that semi-quantum states are exactly those states whose minimal quantum discords vanish [18].

3 Superadditivity for Semi-quantum States

As mentioned in the introduction, it is known that the superadditivity is true for classical states. The following result extends the superadditivity to semi-quantum states.

Proposition 2 *For any semi-quantum state* ρ, *the superadditivity inequality* (1) *holds true.*

Proof By Proposition 1, any semi-quantum state can be represented as

$$\rho = \sum_i p_i P_i \otimes \rho_i,$$

where $P_i = |\psi_i\rangle\langle\psi_i|$ are one-dimensional orthogonal projections, $\{p_i\}$ is a probability distribution, and ρ_i are density operators for party b. The marginal states are

$$\rho_a = \mathrm{tr}_b \rho = \sum_i p_i P_i, \quad \rho_b = \mathrm{tr}_a \rho = \sum_i p_i \rho_i.$$

Note ρ_a has already been written in the spectral decomposition form.

Because $\{|\psi_i\rangle\}$ are orthonormal, we have

$$\rho^\gamma = \sum_i p_i^\gamma P_i \otimes \rho_i^\gamma,$$

from which we obtain

$$[\rho^\gamma, H] = \sum_i p_i^\gamma \big([P_i, H_a] \otimes \rho_i^\gamma + P_i \otimes [\rho_i^\gamma, H_b]\big),$$

where $H = H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b$. Consequently, by noting that $P_i P_j = \delta_{ij} P_i$ and thus $\mathrm{tr}[P_i, H_a]P_j = 0$ for any i, j, we have

$$\begin{aligned}
I_\gamma(\rho, H) &= -\frac{1}{2}\mathrm{tr}[\rho^\gamma, H][\rho^{1-\gamma}, H] \\
&= -\frac{1}{2}\mathrm{tr}\left(\sum_{ij} p_i^\gamma p_j^{1-\gamma} [P_i, H_a][P_j, H_a] \otimes \rho_i^\gamma \rho_j^{1-\gamma}\right) \\
&\quad - \frac{1}{2}\mathrm{tr}\left(\sum_{ij} p_i^\gamma p_j^{1-\gamma} P_i P_j \otimes [\rho_i^\gamma, H_b][\rho_j^{1-\gamma}, H_b]\right) \\
&= -\frac{1}{2}\mathrm{tr}\left(\sum_i p_i [P_i, H_a][P_i, H_a] \otimes \rho_i\right) \\
&\quad - \frac{1}{2}\mathrm{tr}\left(\sum_{i \neq j} p_i^\gamma p_j^{1-\gamma} [P_i, H_a][P_j, H_a] \otimes \rho_i^\gamma \rho_j^{1-\gamma}\right) \\
&\quad + \sum_i p_i I_\gamma(\rho_i, H_b).
\end{aligned}$$

Now we evaluate the above three terms separately. For the first term, we have,

$$\begin{aligned}
&-\frac{1}{2}\mathrm{tr}\left(\sum_i p_i [P_i, H_a][P_i, H_a] \otimes \rho_i\right) \\
&\quad = -\frac{1}{2}\mathrm{tr}\left(\sum_i p_i [P_i, H_a][P_i, H_a]\right)
\end{aligned}$$

$$= \sum_i p_i \big(\langle\psi_i|H_a^2|\psi_i\rangle - \langle\psi_i|H_a|\psi_i\rangle^2\big).$$

For the second term, note that by simple manipulation, for $i \neq j$, we have

$$\mathrm{tr}[P_i, H_a][P_j, H_a] = 2|\langle\psi_i|H_a|\psi_j\rangle|^2,$$

and thus

$$\begin{aligned}
&-\frac{1}{2}\mathrm{tr}\left(\sum_{i\neq j} p_i^\gamma p_j^{1-\gamma}[P_i, H_a][P_j, H_a]\otimes \rho_i^\gamma \rho_j^{1-\gamma}\right)\\
&= -\frac{1}{2}\sum_{i\neq j} p_i^\gamma p_j^{1-\gamma}\big(\mathrm{tr}[P_i, H_a][P_j, H_a]\cdot \mathrm{tr}\rho_i^\gamma \rho_j^{1-\gamma}\big)\\
&= -\sum_{i\neq j} p_i^\gamma p_j^{1-\gamma}\big(|\langle\psi_i|H_a|\psi_j\rangle|^2 \cdot \mathrm{tr}\rho_i^\gamma \rho_j^{1-\gamma}\big)\\
&\geq -\sum_{i\neq j} p_i^\gamma p_j^{1-\gamma}|\langle\psi_i|H_a|\psi_j\rangle|^2,
\end{aligned} \tag{2}$$

where the last inequality follows from the operator Hölder inequality, i.e.,

$$\mathrm{tr}\rho_i^\gamma \rho_j^{1-\gamma} \leq \big(\mathrm{tr}(\rho_i^\gamma)^{1/\gamma}\big)^\gamma \big(\mathrm{tr}(\rho_j^{1-\gamma})^{1/(1-\gamma)}\big)^{1-\gamma} = 1.$$

For the third term, by the convexity of $I_\gamma(\rho, H)$ with respect to ρ, we have

$$\sum_i p_i I_\gamma(\rho_i, H_b) \geq I_\gamma\left(\sum_i p_i\rho_i, H_b\right) = I_\gamma(\rho_b, H_b). \tag{3}$$

Finally, combining inequalities (2) and (3), and noting that

$$\begin{aligned}
I_\gamma(\rho_a, H_a) &= -\frac{1}{2}\mathrm{tr}[\rho_a^\gamma, H_a][\rho_a^{1-\gamma}, H_a]\\
&= \mathrm{tr}\rho_a H_a^2 - \mathrm{tr}\rho_a^\gamma H_a \rho_a^{1-\gamma} H_a\\
&= \sum_i p_i\langle\psi_i|H_a^2|\psi_i\rangle - \sum_{ij} p_i^\gamma p_j^{1-\gamma}|\langle\psi_i|H_a|\psi_j\rangle|^2\\
&= \sum_i p_i\big(\langle\psi_i|H_a^2|\psi_i\rangle - \langle\psi_i|H_a|\psi_i\rangle^2\big) - \sum_{i\neq j} p_i^\gamma p_j^{1-\gamma}|\langle\psi_i|H_a|\psi_j\rangle|^2,
\end{aligned}$$

the desired inequality (1) follows. □

4 Summary

We have classified bipartite states into a hierarchy of classical, semi-quantum, and quantum states. For the intermediate class, i.e., that consists of semi-quantum states, we have established the superadditivity for the Wigner-Yanase-Dyson information. On the other hand, it has been shown that superadditivity always holds true for pure states [15], but cannot be extended to multipartite cases [20]. Consequently, the conditions for the superadditivity seem not directly related to the quantumness of states, but rather more intimately related to the "complexity" of the involved states. Here we regard pure or semi-quantum states as rather simple, and regard truly mixed quantum states with non-orthogonal components as more complex. In this sense, violation of the superadditivity may be interpreted as an indication of the exotic nature of the states.

Since the classical Fisher information is always superadditive [3], the violation of superadditivity in the quantum cases also corroborates the general observation that the informational nature of quantum world is more intricate and subtle than that of the classical world.

The full necessary and sufficient conditions for superadditivity remain to be found.

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